Guided Waves in Moving Media

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Abstract—**This paper contains a theoretical study of the guided waves in a moving isotropic medium. The normal modes which can exist in a circular or rectangular wave guides are found by solving the Maxwell-Minkowski Equations subject to the appropriate boundary conditions. By certain transformations of field vectors, it is possible to change the Maxwell-Minkowski Equations into familiar forms such that the method of vector potentials can be applied to derive complete expressions for the field vectors. The results demonstrate that expressions for the propagation constant and the transverse-wave impedance and admittance in stationary media are modified by terms independent of the guide geometry when the media are moving.**

Introduction

THIS PAPER IS concerned with electromagnetic wave propagation in waveguides, filled with dielectric material, which move down the waveguide with constant velocity. To simplify the treatment, we regard the dielectric as being rigid, so that all portions of the material may be considered to be moving with the same velocity. We further regard the material as being electrically and magnetically homogeneous, isotropic, and linear.

The results of Minkowski's theory are the foundations from which we proceed. Consider two reference frames whose z axes coincide with an appropriate longitudinal element of a waveguide of arbitrary cross section. The primed frame is fixed in the dielectric material, and the unprimed frame is attached to the waveguide (see Fig. 1). Thus, the two reference frames are in relative motion along their z axes.

For the observer at rest in the primed reference frame, the electrodynamics of stationary media apply. In order to find the formulation of electrodynamics which applies for the observer at rest, with respect to the waveguide (i.e., fixed in the unprimed frame), one must cast the electromagnetic field and source vectors into the four-and-six-vector form, and then apply the appropriate form of the Lorentz Transformation according to the special theory of relativity. The field and source quantities transform in such a manner that the form of Maxwell's Equations remains unaltered or invariant. In the unprimed system, Maxwell's Equations are:

$$\nabla \times E + \frac{\partial B}{\partial t} = 0 \qquad (1)$$

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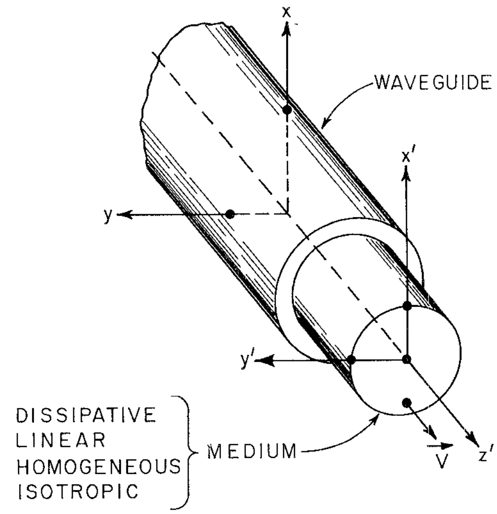

Fig. 1. Circular waveguide filled with a linear, homogeneous, isotropic dissipative medium moving with uniform velocity v with respect to the guide.

$$\nabla \times H - \frac{\partial D}{\partial t} = J \qquad (2)$$

$$\nabla \cdot B = 0 \qquad (3)$$

$$\nabla \cdot D = \rho. \qquad (4)$$

For media at rest, the constitutive relations are known. Thus, in the primed reference frame

$$B' = \mu' H' \qquad (5)$$

$$D' = \epsilon' E' \qquad (6)$$

$$J' = \sigma' E'. \qquad (7)$$

According to Minkowski's theory, (5)–(7) are sufficient for us to determine the constitutive relations for the unprimed field quantities. By applying the Lorentz transformation to the field and current vectors in (5)–(7), Minkowski[1] obtained the following relations

$$D + \frac{v}{c} \times \left(\frac{v}{c} \times D\right) = \epsilon' E + \epsilon' \frac{v}{c_0} \times \left(\frac{v}{c_0} \times E\right) + \left(\frac{1}{c^2} - \frac{1}{c_0^2}\right) v \times H \qquad (8)$$

$$B + \frac{v}{c} \times \left(\frac{v}{c} \times B\right) = \mu' H + \mu' \frac{v}{c_0} \times \left(\frac{v}{c_0} \times H\right) - \left(\frac{1}{c^2} - \frac{1}{c_0^2}\right) v \times E \qquad (9)$$

[1] Sommerfeld, A., *Lectures on Theoretical Physics*, vol III, Electrodynamics. New York: Academic, 1952, pp 280–283.

$$J - \rho v = \frac{\sigma'}{\sqrt{1 - \left(\frac{v}{c_0}\right)^2}} \cdot \left\{ E + v \times B - \frac{v}{c_0}\left[\frac{v}{c_0} \cdot (E + v \times B)\right]\right\} \quad (10)$$

where c and c_0 denote the velocities of light in the medium at rest and in a vacuum (i.e., $c^{-2} = \mu'\epsilon'$ and $c_0^{-2} = \mu_0\epsilon_0$) and v the velocity of the medium (primed frame) relative to the waveguide walls (unprimed frame). If v is assumed to be much smaller than the velocity of light, in the aforementioned relations, terms of second order and higher in v/c or v/c_0 may be neglected when compared with unity. Thus, (8)–(10) become approximately

$$D \approx \epsilon' E + \Lambda \times H \quad (11)$$

$$B \approx \mu' H - \Lambda \times E \quad (12)$$

$$J \approx \rho v + \sigma'(E + v \times B) \quad (13)$$

where $\Lambda \equiv (1/c^2 - 1/c_0^2)v$.

If the Lorentz transformation is applied to the four-current vector $(J, ic_0\rho)$, one finds that

$$\rho' \sqrt{1 - \left(\frac{v}{c_0}\right)^2} = \rho - \frac{v}{c_0^2} \cdot J. \quad (14)$$

If we confine our discussion here to the steady state analysis, the free charge density ρ', associated with the reference frame in which the medium is at rest, vanishes. One notes from (14) that the free charge density measured in the unprimed coordinate system is not zero, but that

$$\rho = \frac{v}{c_0^2} \cdot J. \quad (15)$$

Equations (11)–(13) and (15) can be used to eliminate B, D, J, and ρ from Maxwell's Equations (1)–(4). If the time convention $e^{-i\omega t}$ is used, one may obtain the following equations, which are accurate to the order of v/c:

$$(\nabla + i\omega\Lambda) \times E = i\omega\mu' H \quad (16)$$

$$(\nabla + i\omega\Lambda - \mu'\sigma' v) \times H = -i\omega\epsilon_{\text{eff}}' E \quad (17)$$

$$(\nabla + i\omega\Lambda) \cdot H = 0 \quad (18)$$

$$(\nabla + i\omega\Lambda - \mu'\sigma' v) \cdot E = 0 \quad (19)$$

where

$$\epsilon_{\text{eff}}' = \epsilon'\left(1 + i\frac{\sigma'}{\omega\epsilon'}\right).$$

Equations (16)–(19) might appropriately be designated as "Maxwell-Minkowski Equations for Moving Isotropic Media."

Potential Functions

To find proper solutions of (16)–(19) so as to describe the electromagnetic fields in waveguides, it is convenient to introduce potential functions pertinent to the Maxwell-Minkowski Equations. Similar to the technique used in standard waveguide theory, two types of potential functions, the electric and the magnetic types, will be introduced.

Case I. Electric Type

It will be shown that an electric type vector potential provides a convenient derivation of the field expressions, pertaining to the electric or TM modes. To make use of the standard technique, we first transform (16)–(19) into familiar forms. This can be done by letting

$$E = e^{-i\omega\Lambda\cdot R} E_1 \quad (20)$$

$$H = e^{-i\omega\Lambda\cdot R} H_1 \quad (21)$$

where R denotes the position vector. Substituting (20) and (21) into (16)–(19) one obtains the following equations for E_1 and H_1:

$$\nabla \times E_1 = i\omega\mu' H_1 \quad (22)$$

$$(\nabla - \mu'\sigma' v) \times H_1 = -i\omega\epsilon_{\text{eff}}' E_1 \quad (23)$$

$$\nabla \cdot H_1 = 0 \quad (24)$$

$$(\nabla - \mu'\sigma' v) \cdot E_1 = 0. \quad (25)$$

In view of (24), we may define an electric vector potential function

$$H_1 = \nabla \times A_1. \quad (26)$$

By eliminating H_1 between (22) and (26), we see that an electric scalar potential function can be introduced such that

$$E_1 = i\omega\mu' A_1 - \nabla\phi_1. \quad (27)$$

By substituting the expressions for E_1 and H_1 defined by (26) and (27) into (23), and simplifying the result, we find

$$\nabla^2 A_1 - \mu'\sigma'(v \cdot \nabla)A_1 + k^2 A_1 = \nabla(\nabla \cdot A_1 - \mu'\sigma' v \cdot A_1 - i\omega\epsilon_{\text{eff}}'\phi_1), \quad (28)$$

where

$$k^2 = \omega^2\mu'\epsilon_{\text{eff}}' = \omega^2\mu'\epsilon'\left(1 + i\frac{\sigma'}{\omega\epsilon'}\right). \quad (29)$$

Imposing the condition

$$(\nabla - \mu'\sigma' v) \cdot A_1 - i\omega\epsilon_{\text{eff}}'\phi_1 = 0. \quad (30)$$

Equation (28) then reduces to

$$\nabla^2 A_1 - \mu'\sigma'(v \cdot \nabla)A_1 + k^2 A_1 = 0. \quad (31)$$

If we consider the particular case where A_1 has only a longitudinal component in the z direction, that is,

$$A_1 = \hat{z} A_{1z}, \quad (32)$$

then A_{1z} satisfies the scalar wave equation

$$\nabla^2 A_{1z} - \mu'\sigma' v \cdot \nabla A_{1z} + k^2 A_{1z} = 0. \quad (33)$$

It can easily be shown by means of (25), (27), and (30), that the scalar potential function ϕ_1 satisfies the wave equation defined by (33). It will be shown later that the solutions for A_{1z}, obtained under the appropriate boundary conditions, give us a complete set of TM modes in a cylindrical waveguide. The explicit relations between A_{1z} and the field vector are:

$$H^{(e)} = e^{-i\omega\Lambda \cdot R} \nabla \times [A_{1z}\hat{z}] \quad (34)$$

$$E^{(e)} = \frac{1}{-i\omega\epsilon_{\text{eff}}'} e^{-i\omega\Lambda \cdot R}(\nabla - \mu'\sigma' v) \times \nabla \times (A_{1z}\hat{z}) \quad (35)$$

where the subscript "e" denotes the fields of electric type or transverse-magnetic.

Case II. Magnetic Type

To introduce the magnetic type vector potential let

$$E = \exp[(-i\omega\Lambda + \mu'\sigma' v) \cdot R] E_2 \quad (36)$$

$$H = \exp[(-i\omega\Lambda + \mu'\sigma' v) \cdot R] H_2. \quad (37)$$

Equations (16)–(19) become

$$(\nabla + \mu'\sigma' v) \times E_2 = i\omega\mu' H_2 \quad (38)$$

$$\nabla \times H_2 = -i\omega\epsilon_{\text{eff}}' E_2 \quad (39)$$

$$(\nabla + \mu'\sigma' v) \cdot H_2 = 0 \quad (40)$$

$$\nabla \cdot E_2 = 0. \quad (41)$$

In view of (41), we may introduce a magnetic type vector potential A_2

$$E_2 = \nabla \times A_2. \quad (42)$$

The remaining steps are similar to the TM case. Thus, if A_2 has only a longitudinal component A_{2z}, then A_{2z} satisfies the wave equation

$$\nabla^2 A_{2z} \times \mu'\sigma' v \cdot \nabla A_{2z} + k^2 A_{2z} = 0. \quad (43)$$

Solutions of (43), obtained under the proper boundary conditions, gives us the complete set of magnetic or TE modes in a cylindrical waveguide. The field vectors are related to A_{2z} by

$$E^{(m)} = \exp[(i\omega\Lambda + \mu'\sigma' v) \cdot R] \nabla \times (A_{2z}\hat{z}) \quad (44)$$

$$H^{(m)} = \frac{1}{i\omega\mu'} \exp[(i\omega\Lambda + \mu'\sigma' v) \cdot R](\nabla + \mu'\sigma' v) \times \nabla \times (A_{2z}\hat{z}). \quad (45)$$

It should be observed that (33) and (43) defined for A_{1z} and A_{2z} are *not* identical.

TM Modes in a Circular Cylindrical Waveguide

In this section, we shall give a detailed derivation of the wave functions pertaining to the TM modes in a circular cylindrical waveguide, whose axis coincides with the z axis (Fig. 1). The medium inside the guide is moving along the guide with a constant velocity

$$v = v\hat{z}. \quad (46)$$

Applying the method of separation of variables in a cylindrical coordinate system to (33) gives

$$A_{1z} = AJ_n(k^e r) \begin{matrix} \cos \\ \sin \end{matrix} n\phi e^{i\gamma^e z} \quad (47)$$

where the constants k^e and γ^e satisfy

$$(\gamma^e)^2 + i\mu'\sigma' v\gamma^e - k^2 + (k^e)^2 = 0. \quad (48)$$

For a perfectly conducting waveguide, which is assumed here, the electric field satisfies the same boundary condition as in the case of stationary media

$$\hat{r} \times E^e = 0 \quad \text{at} \quad r = a, \quad (49)$$

where a denotes the radius of the guide. From (35), it can be shown that both the ϕ and z components of $E^{(e)}$ are proportional to $J_n(k^e r)$. The constant k^e, as in the ordinary theory of waveguide, is therefore determined by the roots of

$$J_n(k_{nl}^e a) = 0, \quad (50)$$

where the subscripts n and l denote, respectively, the order of the Bessel Function and the index of the root.[2] If the roots of the Bessel Functions are denoted by p_{nl}, then

$$k_{nl}^e = p_{nl}/a. \quad (51)$$

When the quadratic equation (48) is solved for γ^e one obtains, after neglecting terms of the order of $(v/c)^2$,

$$\gamma_{nl}^e = \pm [k^2 - (k_{nl}^e)^2]^{1/2} - \frac{i}{2}\mu'\sigma' v. \quad (52)$$

Finally, by substituting A_{1z} defined by (47) into (34) and (35), one obtains the complete expressions for the electric and magnetic field vectors.

$$H_r^e = \mp A \frac{n}{r} J_n(k_{nl}^e r) \begin{matrix} \sin \\ \cos \end{matrix} n\phi e^{i\Gamma_{nl}^e z} \quad (53)$$

$$H_\phi^e = -A \frac{\partial J_n}{\partial r} \begin{matrix} \cos \\ \sin \end{matrix} n\phi e^{i\Gamma_{nl}^e z} \quad (54)$$

$$E_t^e = \text{transverse part of } E^e = -Z_{nl}^e[\hat{z} \times H^e] \quad (55)$$

$$E_z^e = \frac{(k_{nl}^e)^2}{-i\omega\epsilon_{\text{eff}}'} AJ_n(k_{nl}^e r) \begin{matrix} \cos \\ \sin \end{matrix} n\phi e^{i\Gamma_{nl}^e z}, \quad (56)$$

where

$$\Gamma_{nl}^e = \gamma_{nl}^e - \omega_\Lambda. \quad (57)$$

[2] The nomenclature concerning the index of the root is discussed in: Tai, C. T., On the nomenclature of TE_{0l} modes in a cylindrical waveguide, *Proc. IRE* (*Correspondence*), vol 49, Sep 1961, p 1442.

The term ω_A, according to the explanation following (13), is given by

$$\omega_A = \frac{\omega v}{c^2}\left(1 - \frac{c^2}{c_0^2}\right) = \frac{\omega v}{c^2}\tau. \tag{58}$$

Hence, using (52), (57) may be written in the form

$$\Gamma_{nl}^e = \pm [k^2 - (k_{nl}^e)^2]^{1/2} - \frac{\omega v}{c^2}\left(\tau + i\frac{\sigma'}{2\omega\epsilon'}\right). \tag{59}$$

The first term, on the right of (59), is the propagation constant of the guided wave, when the medium is stationary. The correction term due to the motion of the medium is seen to be independent of the dimension of the guide. This characteristic can be shown to be valid for guides of any arbitrary cross section.

The constant Z_{nl}^e in (55) is designated as the transverse-wave impedance pertaining to the TM modes. It is defined by

$$Z_{nl}^e = \frac{\gamma_{nl}^e + i\mu'\sigma' v}{\omega\epsilon'} = \frac{1}{\omega\epsilon'}\left\{\pm [k^2 - (k_{nl}^e)^2]^{1/2} + i\frac{\mu'\sigma' v}{2}\right\}. \tag{60}$$

Again, (60) reduces to the ordinary wave impedance when the medium is stationary. This completes our derivation for the TM modes.

TE Modes in a Circular Cylindrical Waveguide

The derivation of the expressions for A_{2z}, and the corresponding electric and magnetic field vectors, is almost the same as for the TM case. We give here only the results.

$$A_{2z} = BJ_n(k_{nl}^m r) \begin{matrix}\cos\\ \sin\end{matrix}\, n\phi e^{i\gamma_{nl}^m z}, \tag{61}$$

where

$$k_{nl}^m = p_{nl}'/a, \tag{62}$$

and p_{nl}' denotes the roots of the derivative of the Bessel Functions. The constant γ_{nl}^m, correct to the order of v/c, is given by

$$\gamma_{nl}^m = \pm [k^2 - (k_{nl}^m)^2]^{1/2} + i\frac{\mu'\sigma' v}{2}. \tag{63}$$

The expressions for the electric and magnetic field vectors are:

$$E_r^m = \mp B\frac{n}{r}J_n(k_{nl}^m r) \begin{matrix}\sin\\ \cos\end{matrix}\, n\phi e^{i\Gamma_{nl}^m z} \tag{64}$$

$$E_\phi^m = -B\frac{\partial J_n}{\partial r} \begin{matrix}\cos\\ \sin\end{matrix}\, n\phi e^{i\Gamma_{nl}^m z} \tag{65}$$

$$\begin{aligned} H_t^m &= \text{transverse part of } H^m \\ &= Y_{nl}^m(\hat{z} \times E^m) \end{aligned} \tag{66}$$

$$H_z^m = \frac{(k_{nl}^m)^2}{i\omega\mu'}BJ_n(k_{nl}^m r) \begin{matrix}\cos\\ \sin\end{matrix}\, n\phi e^{i\Gamma_{nl}^m z}, \tag{67}$$

where

$$\Gamma_{nl}^m = \pm [k^2 - (k_{nl}^m)^2]^{1/2} - \frac{\omega v}{c^2}\left(\tau + i\frac{\sigma'}{2\omega\epsilon'}\right), \tag{68}$$

$$Y_{nl}^m = \frac{1}{\omega\mu'}(\gamma_{nl}^m - i\mu'\sigma' v) = \frac{1}{\omega\mu'}\left\{\pm [k^2 - (k_{nl}^m)^2]^{1/2} - i\frac{\mu'\sigma' v}{2}\right\}. \tag{69}$$

The Rectangular Waveguide

The fields and the propagation constant for a waveguide of rectangular cross section are found by the same procedure used for the circular waveguide. The complete expressions for the electric and magnetic fields are given below without derivation. The dimensions of the guide and the coordinate system used are illustrated in Fig. 2.

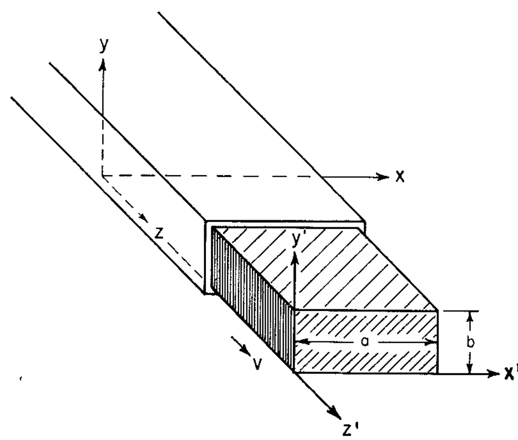

Fig. 2. Rectangular waveguide filled with a moving medium.

Case I. TM Modes

$$H_x^e = A\frac{n\pi}{b}\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{i\Gamma_{mn}z} \tag{70}$$

$$H_y^e = -A\frac{m\pi}{a}\cos\left(\frac{m\pi}{a}\right)x\sin\left(\frac{n\pi}{b}y\right)e^{i\Gamma_{mn}z} \tag{71}$$

$$E_t^e = -Z_{mn}^e(\hat{z} \times H^e) \tag{72}$$

$$E_z^e = A\frac{k_{mn}^2}{-i\omega\epsilon'_{\text{eff}}}\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{i\Gamma_{mn}z}, \tag{73}$$

where

$$k_{mn}^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$

$$\Gamma_{mn} = \pm (k^2 - k_{mn}^2)^{1/2} - \frac{\omega v}{c^2}\left(\tau + i\frac{\sigma'}{2\omega\epsilon'}\right) \tag{74}$$

$$k^2 = \omega^2\mu'\epsilon'_{\text{eff}} = \omega^2\mu'\epsilon'\left(1 + i\frac{\sigma'}{\omega\epsilon'}\right)$$

$$\tau = 1 - \frac{c^2}{c_0^2} = 1 - \frac{\mu_0\epsilon_0}{\mu'\epsilon'}$$

$$Z_{mn}{}^e = \frac{1}{\omega\epsilon'}\left[\pm(k^2 - k_{mn}{}^2)^{1/2} + i\frac{\mu'\sigma'v}{2}\right]. \qquad (75)$$

Case II. TE Modes

$$E_x{}^m = -B\frac{n\pi}{b}\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{i\Gamma_{mn}z} \qquad (76)$$

$$E_y{}^m = B\frac{m\pi}{a}\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{i\Gamma_{mn}z} \qquad (77)$$

$$\boldsymbol{H}_t{}^m = Y_{mn}{}^m(\hat{z} \times \boldsymbol{E}^m) \qquad (78)$$

$$H_z{}^m = \frac{k_{mn}{}^2}{i\omega\mu'}\cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{i\Gamma_{mn}z}, \qquad (79)$$

where

$$Y_{mn}{}^m = \frac{1}{\omega\mu'}\left[\pm(k^2 - k_{mn}{}^2)^{1/2} - i\frac{\mu'\sigma'v}{2}\right]. \qquad (80)$$

The other constants occurring in (75)–(79) are defined by (74).

Conclusion

The authors felt it worthwhile to include some of Minkowski's work which led to (16)–(19). The authors are convinced of the futility of trying to describe constitutive parameters μ, ϵ and σ of media in motion. As Minkowski realized, only those parameters in a medium at rest μ', ϵ' and σ', have physical meaning.

With the aid of the Maxwell-Minkowski Equations, we have derived and solved the wave equations for the electric or the magnetic field, pertaining to the guided waves in a circular or rectangular waveguide. The solution is facilitated by the introduction of vector and scalar potential-functions associated with this problem. The results demonstrate that for a moving medium, the fields, to the first-order of v/c differ in only two respects from the fields obtained when the medium is at rest. First, the propagation constant is modified by a term which depends upon the velocity as well as the constitutive constants of the stationary media, but is independent of the cross section of the guide; second, the transverse-wave impedance or admittance is also modified by a term which is independent of the dimension of the guide.

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Generalized Solutions for Optical Maser Amplifiers

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Abstract—The optical maser amplifier is treated from the transient analysis point of view using the Laplace transform method as opposed to the conventional sinusoidal steady-state analysis that sometimes leads to inconsistent results especially for the region beyond threshold. Firstly, the wave equations are expressed in terms of Laplace transforms, and then the generalized solutions for both the transmission and the reflection mode of operation are derived taking the transient terms into account. Finally, the inverse Laplace transforms are obtained yielding the generalized solutions in terms of real-time functions. In order to emphasize the point of the argument and also to compare the results of the usual sinusoidal steady-state analysis, use is made of the simplest possible model of a one-dimensional system consisting of three media, air, active medium, and air. An incident coherent transverse electromagnetic wave, which falls normally on the surface of the system, is assumed. The generalized solutions derived agree, in the region below threshold, exactly with that of the sinusoidal steady-state analysis obtained previously by other investigators. However, for the region beyond critical threshold, the generalized solutions indicate that the device goes into a state of self-oscillation with oscillation frequencies that strictly coincide with those of the Fabry-Perot type resonator. Thus, the limitation of applicability of the conventional sinusoidal steady-state analysis is clarified. Some remarks are also given on the design problem of optical maser amplifiers in connection with the transient terms involved.

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Introduction

To the authors' knowledge, most of the theoretical treatments of an optical maser amplifier reported so far have been based on the sinusoidal steady-state analysis. The investigations by Jacobs, et al. [1], [2] are typical of those approaches in which the optical maser amplifier is treated as a transmission-line or boundary-value problem in electro-